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FUNDAMENTAL GENERAL RELATIONSHIPS FOR A MODEL OF AN ISOTROPICALLY ELASTIC HETEROMODULAR MEDIUM†

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A problem of non-smooth mechanics, the expansion of the stress potential for heteromodular media with respect to the components of the strain tensor, is formulated and solved. New characteristic relationships and the form of the functional dependence of the elastic moduli and the phase of the similitude of the deviators on the relationship between the invariants of the strain tensor follow from the proposed expression. Inversion of governing dependences of the stresses on the deformations is carried out. The nature of the coupling between the generalized moduli of elasticity and the phase of the similitude of the deviators during deformations is demonstrated. Dual formulations of the theory and the energy principles are presented.

1. THE ASYMMETRY AND NON-SMOOTHNESS OF THE HETEROMODULAR LAW OF ELASTICITY

IN THE case of the simplest heteromodular law of elasticity [1–4]

$$\sigma=\begin{cases}E^{-}\varepsilon, & \varepsilon\leqslant 0\\ E^{+}\varepsilon, & \varepsilon\geqslant 0\end{cases}\tag{1.1}$$

the stress potential, that is, the specific potential energy of strain, has the form (E^+ and E^- are the elastic moduli under compression and extension, respectively)

$$U=\begin{cases}\tfrac{1}{2}E^{-}\varepsilon^2, & \varepsilon\leqslant 0\\ \tfrac{1}{2}E^{+}\varepsilon^2, & \varepsilon\geqslant 0\end{cases}\tag{1.2}$$

It is obvious that relationship (1.1) is not smooth at zero and that the potential $U(\varepsilon)$ (1.2) is not analytic and that it is not an even function.

In the general case we assume that the elastic potential $W(\varepsilon)$ for isotropic, heteromodular media (HMM) is a homogeneous function of the second degree of homogeneity with respect to the components of the strain tensor

$$\varepsilon=(\varepsilon_{ij}),\quad W(0)=0,\quad W(\varepsilon)\geqslant\kappa\varepsilon_{ij}\varepsilon_{ij},\quad \kappa=\text{const}>0$$

Such a potential has been considered in [4] under the assumption that the tensor is linear.

As a consequence of the effects of heteromodularity, the potential $W(\varepsilon)$ is also not an analytic function at zero and its symmetry group cannot include inversion.

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Actually, as a consequence of Euler's theorem on homogeneous functions, we have

$$W(\varepsilon) = \tfrac{1}{2} Q_{ij}\varepsilon_i\varepsilon_j, \quad Q_{ij} = \frac{\partial^2 W}{\partial\varepsilon_i \partial\varepsilon_j} (\det\|\varepsilon_{ij} - \varepsilon_k\delta_{ij}\| = 0; \; i,j,k = 1,2,3) \tag{1.3}$$

The coefficients Q_{ij} are homogeneous functions of zero degree of homogeneity which are determined by the values of the generalized moduli of elasticity and the phase of the similitude of the deviators. In the first place, since $Q_{ij}(\varepsilon) \neq Q_{ij}(-\varepsilon)$, according to the definition of a HMM, then, consequently, the function $W(\varepsilon)$ cannot be expanded in a Taylor series at the point $\varepsilon = 0$. Similarly, according to (1.3), $W(\varepsilon) \neq W(-\varepsilon)$, that is, the function $W(\varepsilon)$ does not have a centre of inversion.

The asymmetry of isotropic HMMs admits [5] of the possibility of the existence of two different versions of these which have opposite heteromodularity signs, that is, opposite signs for the difference in the corresponding elastic moduli and the phases of the similitude of the deviators. Here, media with different absolute values of these differences can have mirror-symmetry properties while being the antipodes of one another. A mixture containing equal amounts of the antipodes is indistinguishable from a conventional elastic body.

Models of HMMs usually arise when weakly non-linear strain diagrams are replaced by piecewise linear diagrams. In this case, breakdown of the closeness of the solutions of problems corresponding to the smooth and uneven (heteromodular) models does not occur [6]. In the general case, non-linear strain diagrams can be approximated by a broken line. This enables one to reduce an initial non-linear problem corresponding to a smooth model to a set of problems in the mechanics of HMMs, the solutions of which must be joined together.

The potential $W(\varepsilon)$ is a piecewise-analytic function and cannot be expanded in a power series. Assuming tensor linearity, this potential has been directly determined in [4, 7] on the basis of experimental strain diagrams obtained under proportional loads.

2. EXPANSION OF THE STRESS POTENTIAL OF AN HMM, $W(\varepsilon)$ IN SERIES

The symmetry group of the potential $W(\varepsilon)$ is identical with a proper rotation group. Let us represent the potential $W(\varepsilon)$ in the form of a series in functions which form a basis of representations of this group. By virtue of (1.3), we have

$$W(\varepsilon) = r^2 F(\varepsilon_1/r, \varepsilon_2/r, \varepsilon_3/r) \tag{2.1}$$

The function F is defined on the surface of a sphere of radius $r = |\varepsilon| = \sqrt{(\varepsilon_{ij}\varepsilon_{ij})} = 1$ with its centre at the point (0, 0, 0) in the space of the principal strains.

Surface spherical functions form the basis of representations of the spatial rotation group. It is obvious that F is a function with an integrable square of the modulus on the surface of the given sphere. Then [8], this function can be expanded in series, which converges in the mean, in an orthogonal system of surface spherical functions Y_l^m

$$F\left(\frac{\varepsilon_1}{r}, \frac{\varepsilon_2}{r}, \frac{\varepsilon_3}{r}\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^{(m)} Y_l^m \tag{2.2}$$

In a canonical local system of coordinates, the axes of which coincide with the principal axes of the strain tensor, the functions Y_l^m can be expressed in terms of the principal components of this tensor.

In fact, by using the integral representation for the spherical functions

$$Y_l^m = \frac{(-\mathrm{i})^m (l+m)!}{2\pi r^l l!} \int_{-\pi}^{\pi} X^l \cos mu\, du, \quad m = 0, 1, \ldots, l$$

$$Y_l^m = -\frac{i^m(l-m)!}{2\pi r^l l!}\int_{-\pi}^{\pi} X^l \sin mu du, \quad m=-1,-2,\ldots,-l$$
$$X = \varepsilon_3 + i\varepsilon_1\cos u + i\varepsilon_2 \sin u; \quad (i^2=-1)$$

which follow from the general form of the solutions of Laplace's equation [9], expansion (2.2) can be written in the form

$$F\left(\frac{\varepsilon_1}{r},\frac{\varepsilon_2}{r},\frac{\varepsilon_3}{r}\right) = c_0 + c_i\frac{\varepsilon_i}{r} + c_{ij}\frac{\varepsilon_i\varepsilon_j}{r^2} + c_{ijk}\frac{\varepsilon_i\varepsilon_j\varepsilon_k}{r^3} + \ldots \tag{2.3}$$
$$c_0 = a_0^{(0)}, \quad c_1 = a_1^{(1)}, \quad c_2 = a_1^{(-1)}, \quad c_3 = a_1^{(0)}$$
$$c_{12} = c_{21} = 3a_2^{(-2)}, \quad c_{23} = c_{32} = \tfrac{3}{2}a_2^{(-1)}, \quad c_{13} = c_{31} = \tfrac{3}{2}a_2^{(1)}$$
$$c_{11} = 3a_2^{(2)} - \tfrac{1}{2}a_2^{(0)}, \quad c_{22} = -3a_2^{(2)} - \tfrac{1}{2}a_2^{(0)}, \quad c_{33} = a_2^{(0)}$$
$$c_{111} = -\tfrac{3}{2}a_3^{(1)} + 15a_3^{(3)}, \quad c_{222} = -\tfrac{3}{2}a_3^{(-1)} - 15a_3^{(-3)}, \quad c_{333} = a_3^{(0)}$$
$$c_{122} = -\tfrac{3}{2}a_3^{(1)} - 45a_3^{(3)}, \quad c_{133} = 6a_3^{(1)}, \quad c_{211} = -\tfrac{3}{2}a_3^{(-1)} + 45a_3^{(-3)}$$
$$c_{233} = 6a_3^{(-1)}, \quad c_{311} = -\tfrac{3}{2}a_3^{(0)} + 15a_3^{(2)}, \quad c_{322} = -\tfrac{3}{2}a_3^{(0)} - 15a_3^{(2)}$$
$$c_{123} = c_{132} = c_{213} = c_{231} = c_{312} = c_{321} = 5a_3^{(-2)}$$

Expansion (2.3) is invariant under rotations. According to the transformation rule $\varepsilon_k = q_{kp}^2\varepsilon_p'$, the coefficients of the series in (2.3) satisfy the equalities

$$c_p' = q_{kp}^2 c_k, \quad c_{pt}' = q_{kp}^2 q_{st}^2 c_{ks}, \quad c_{pr\upsilon}' = q_{kp}^2 q_{sr}^2 q_{u\upsilon}^2 c_{ksu}; \quad q_{ij} = \cos(\mathbf{l}_i, \mathbf{l}_j') \tag{2.4}$$

In these relationships, $\mathbf{l}_i$, $\mathbf{l}_j'$ are the vectors of the orthonormalized basis of the initial and the new canonical coordinate systems respectively. On rotating the canonical system of coordinates through a right angle in an anticlockwise sense around the ε_1, ε_2 and ε_3 axes, we have

$$q_{kp} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad q_{kp} = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{Vmatrix}, \quad q_{kp} = \begin{Vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

In this case, we obtain

$$c_0 \equiv B_0, \quad c_1 = c_2 = c_3 \equiv B_1, \quad c_{11} = c_{22} = c_{33} \equiv 0$$
$$c_{12} = c_{13} = c_{23} \equiv B_2, \quad c_{111} = c_{222} = c_{333} \equiv B_3$$
$$c_{122} = c_{133} = c_{211} = c_{233} = c_{311} = c_{322} \equiv -\tfrac{3}{2}B_3, \quad c_{123} \equiv B_4$$

Hence, expansion (2.3) acquires the form

$$F(\varepsilon_1/r, \varepsilon_2/r, \varepsilon_3/r) = B_0 - B_2 + (B_1 - \tfrac{3}{2}B_3 - 3B_4)\xi + \\ + B_2\xi^2 + B_4\xi^3 + (\tfrac{5}{2}B_3 + 2B_4)\eta^3 + \ldots \tag{2.5}$$

The invariance of the potential $W(\varepsilon)$ with respect to inversion is attributable to the terms in series (2.5) which depend on the dimensionless scalar parameters ξ and η

$$\xi = I_1/\sqrt{I_2}, \quad |\varepsilon| \leq \sqrt{3}; \quad \eta = I_3^{1/3}/\sqrt{I_2}, \quad |\eta| \leq 1$$
$$(I_1 = \varepsilon_{ij}\delta_{ij}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki})$$

The parameter ξ characterizes the relationship between the bulk and the mean shear strains in an elementary volume of the medium while the parameter η additionally takes account of the relationship between the maximum and mean shears. Where necessary, analogous parameters of one of them are introduced into the treatment in the analysis of experimental strain diagrams of materials, the characteristics of which depend on the form of the stressed state [10, 11].

The terms of the series, corresponding to $l=4, 5, \ldots$ from (2.2), which have not been written out in expansion (2.5), express the finer details of the change in the function F depending on the form of the strain, that is, on the values of the parameters ξ and η. Confining ourselves to the terms of the series in (2.5) and introducing the appropriate notation

$$B_2 = \tfrac{1}{2}\lambda, \quad B_0 = \mu - \tfrac{1}{2}\lambda, \quad B_4 = \alpha, \quad B_3 = \tfrac{2}{5}(\beta - 2\alpha), \quad B_1 = \tfrac{3}{5}(\beta + 3\alpha) - \nu$$

we obtain

$$W(\varepsilon) = \frac{\lambda}{2} I_1^2 + \mu I_2 - \nu I_1\sqrt{I_2} + \alpha\frac{I_1^3}{\sqrt{I_2}} + \beta\frac{I_3}{\sqrt{I_2}}, \quad |\varepsilon| > 0 \tag{2.6}$$

It is clear that, if we require that the expression for $W(\varepsilon)$ should be invariant under inversion $\varepsilon_i' = -\varepsilon_i$, then $W(\varepsilon)$ becomes the classical elastic potential. The heteromodularity is described by the terms in expression (2.6), starting with the third term, which may be considered as a first approximation [7, 12, 13].

The realization of the potential $W(\varepsilon)$ in the form (2.6) using expansion (2.3) of the function $F(\varepsilon_1/r, \varepsilon_2/r, \varepsilon_3/r)$ can be considered as a basis for carrying out systematic investigations of the effects of heteromodularity in isotropically elastic media.

3. IMPORTANT RELATIONSHIPS IN THE MECHANICS OF HMMS

Using expression (2.6) and the conditions imposed on $W(\varepsilon)$, the governing equations relating the stresses to the strains can be written as follows:

$$\begin{aligned} &\sigma_{ij} = (\lambda - \nu/\xi + 3\alpha\xi)I_1\delta_{ij} + (2\mu - \nu\xi - \alpha\xi^3 - \beta\eta^3)\varepsilon_{ij} + 3\beta\varepsilon_{ik}\varepsilon_{jk}/\sqrt{I_2}, \\ &|\varepsilon| > 0; \quad \sigma_{ij} = 0, \quad |\varepsilon| = 0 \end{aligned} \tag{3.1}$$

In (3.1), the stress and strain tensors are coaxial.

For the inversion of (3.1), we make use of the general form of the expansion of the strain tensor in the basis of the space of the coaxial stress tensors [14–16]

$$\varepsilon_{ij} = \frac{1}{3} I_1 \frac{\partial J_1}{\partial \sigma_{ij}} + \tau\gamma\cos\chi\left(\frac{1}{\tau}\frac{\partial\tau}{\partial\sigma_{ij}} - \mathrm{tg}\,\chi\frac{\partial\chi_\sigma}{\partial\sigma_{ij}}\right) \tag{3.2}$$

Expressions for the generalized bulk modulus $K + \varphi(\xi, \eta)$, shear modulus $G + \psi(\xi, \eta)$ and phase of the similitude of the deviators $\chi(\xi, \eta)$ can be obtained by convolution of the product of relationships (3.1) with the tensors $\delta = (\delta_{ij})$, $\sigma = (\sigma_{ij})$ and $\mathrm{tr}\sigma^2$, respectively. As a result taking account of the equality

$$2W = \tfrac{1}{3} I_1 J_1 + \tau\gamma\cos\chi$$

and the expression for the trace of the power of the strain tensor using the Hamilton–Cayley theorem, we obtain

$$K+\varphi(\xi,\eta)=\lambda+\tfrac{2}{3}\mu-\nu\left(\frac{1}{\xi}+\frac{\xi}{3}\right)+\alpha\left(3\xi-\frac{\xi^3}{3}\right)+\beta\left(\frac{1}{\xi}-\frac{\eta^3}{3}\right)$$

$$G+\psi(\xi,\eta)=\left[g^2(\xi,\eta)+\frac{9\beta^2}{3-\xi^2}(\tfrac{1}{8}-\tfrac{3}{4}\xi^2+\tfrac{1}{8}\xi^4+\xi\eta^3)+3\beta\frac{3\eta^3-\xi}{3-\xi^2}g(\xi,\eta)\right]^{1/2} \tag{3.3}$$

$$\chi(\xi,\eta)=\arccos\left\{[G+\psi(\xi,\eta)]^{-1}\left(\mu-\tfrac{1}{2}\nu\xi-\tfrac{1}{2}\alpha\xi^3-\tfrac{1}{2}\beta\frac{\xi-2\eta^3-\frac{1}{3}\xi^2\eta^3}{1-\frac{1}{3}\xi^2}\right)\right\}$$

Hence, the governing equations relating the strains and the stresses are finally written in the following form

$$\varepsilon_{ij}=\frac{\frac{1}{9}J_1\delta_{ij}}{K+\varphi(\xi,\eta)}+\frac{\frac{1}{2}}{G+\psi(\xi,\eta)}\left[(\cos\chi+\sin\chi\,\mathrm{tg}\,3\chi_\sigma)s_{ij}+\frac{3\sin\chi}{\cos 3\chi_\sigma}\left(\frac{s_{ik}s_{kj}}{\tau}-\frac{2}{9}\tau\delta_{ij}\right)\right],$$

$$|\sigma|>0;\quad \varepsilon_{ij}=0,\quad |\sigma|=0\quad (|\sigma|=\sqrt{\sigma_{ij}\sigma_{ij}}) \tag{3.4}$$

In relationships (3.4), the generalized moduli and phase of the similitude of the deviators are defined by (3.3) and by the relationships

$$J_1=\sigma_{ij}\delta_{ij},\quad \tau=\sqrt{\tfrac{3}{2}s_{ij}s_{ij}},\quad \gamma=\sqrt{\tfrac{2}{3}e_{ij}e_{ij}},\quad s_{ij}=\sigma_{ij}-\tfrac{1}{3}J_1\delta_{ij},\quad e_{ij}=\varepsilon_{ij}-\tfrac{1}{3}I_1\delta_{ij}$$

$$\chi=\chi_\sigma-\chi_\varepsilon,\quad \chi_\sigma=-\tfrac{1}{3}\arcsin(\tfrac{9}{2}S_3/\tau^3),\quad |\chi_\sigma|\leq\pi/6 \tag{3.5}$$

$$\chi_\varepsilon=-\tfrac{1}{3}\arcsin(\tfrac{4}{3}E_3/\gamma^3),\quad |\chi_\varepsilon|\leq\pi/6,\quad S_3=s_{ij}s_{jk}s_{ki},\quad E_3=e_{ij}e_{jk}e_{ki}$$

$$g(\xi,\eta)=\mu-\tfrac{1}{2}\nu\xi-\tfrac{1}{2}\alpha\xi^3-\tfrac{1}{2}\beta\eta^3$$

The values of the parameters ξ and η are found from the system of equations

$$\frac{3\xi[K+\varphi(\xi,\eta)]}{H(\xi,\eta)}=\frac{J_1}{\tau},\quad \frac{(\eta^3-\xi+\frac{2}{9}\xi^3)\theta(\xi,\eta)}{H^3(\xi,\eta)}=\frac{S_3}{\tau^3} \tag{3.6}$$

Here

$$H(\xi,\eta)=\sqrt{2(3-\xi^2)}[G+\psi(\xi,\eta)]$$

$$\theta(\xi,\eta)=8g^3(\xi,\eta)+(\eta^3-\xi+\tfrac{2}{9}\xi^3)^{-1}\{\tfrac{3}{4}\beta^3[12\eta^6-12\xi\eta^3(1-\xi^2)-$$
$$-1+9\xi^2-15\xi^4+3\xi^6]+2\beta g^2(\xi,\eta)(12\xi\eta^3+3-14\xi^2+3\xi^4)+$$
$$+\beta^2 g(\xi,\eta)[3\eta^3(3+7\xi^2)+3\xi-27\xi^3+6\xi^5]\}$$

In the case of plane strain, the parameters ξ and η are inter-related $\eta^3=\frac{1}{2}\xi(3-\xi^2)$, and the solution of system (3.6) reduces to the solution of just a single equation of this system.

In the general case, according to the last relationship of (3.3), the deviators of the stress and strain tensors are not similar in that the phase of their similitude χ when $\beta\neq 0$ is non-zero and the ratios of the principal components of the deviators of these tensors are not equal to one another. In the case of a uniform triaxial compression or extension $\tau=0$, $S_3=0$, shear strains do not arise in the medium according to (3.6): $\xi=\mp\sqrt{3}$, $\eta^3=\mp 1/\sqrt{3}$. In the case of pure shear $J_1=0$, shear strains are accompanied by bulk strains. Under simple shear Poynting and Kelvin–Wertheim effects occur. In the case of a proportional change in the stresses $\sigma_{ij}=t\sigma_{ij}^0$, $t=\mathrm{const}$, the parameters ξ and η remain constant and, consequently, the strain diagrams (3.4) are rectilinear.

The potential $W(\varepsilon)$ can be considered as a function of the state parameters I_1, γ and χ_ε. The differential relationships, relating the generalized moduli of elasticity and the phase of the similitude of the deviators are the condition for this

$$3\xi^2\eta^2\varphi'_\xi + 2\eta^2(3-\xi^2)[\cos\chi(G+\chi)]'_\xi + (1-\xi\eta^3)\left\{\frac{3\xi^2}{3-\xi^2}\varphi'_\eta + 2[\cos\chi(G+\psi)]'_\eta\right\} = 0$$

$$4\eta^2(3-\xi^2)[\sin\chi(G+\psi)]'_\xi + \xi\sqrt{2(3-\xi^2)}\cos 3\chi_\varepsilon\varphi'_\eta + 4(1-\xi\eta^3)[\sin\chi(G+\psi)]'_\eta = 0$$

$$6\xi\eta^2(3-\xi^2)[\sin\chi(G+\psi)]'_\xi = \sqrt{2}(3-\xi^2)^{3/2}\cos 3\chi_\varepsilon[\cos\chi(G+\psi)]'_\eta + \tag{3.7}$$

$$+3\xi\frac{\xi\eta^3-1}{3-\xi^2}[\sin\chi(G+\psi)]'_\eta + 36\eta^3\sin\chi(G+\psi)$$

Relationships (3.7) determine the nature of the interconnected change in the moduli and phase when there is a change in the parameters ξ and η during the strain. According to (3.7), when there is a proportional change in the components of the stress tensor, the phase of the similitude of the deviators χ is equal to zero and the governing relationships (3.4) and (3.1) are linear. The parameter η can only be constant when there is a proportional change in the stresses. If the generalized bulk modulus is an increasing function of η, the product of the generalized shear modulus with $\cos\chi$ is a decreasing function of η and $\chi_\sigma > \chi_\varepsilon$ and vice versa. We note that $\operatorname{sign}\varphi'_\eta = -\operatorname{sign}\beta$. Relationships (3.7) also determine the possible forms of the strain diagrams of HMMs.

4. DUAL FORMULATIONS. ENERGY PRINCIPLES FOR HMMS

The equalities

$$W(\varepsilon) + W^*(\sigma) = \sigma_{ij}\varepsilon_{ij}, \quad W^*(\sigma) = \sup_\varepsilon[\varepsilon_{ij}\sigma_{ij} - W(\varepsilon)] \tag{4.1}$$

hold in the case of the potential $W(\varepsilon)$.

Here, by virtue of Euler's theorem

$$W(\varepsilon) = W^*(\sigma) = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{4.2}$$

The Legendre transformation [6] of the potential $W(\varepsilon)$ is involute

$$W(\varepsilon) = \sup_\sigma[\sigma_{ij}\varepsilon_{ij} - W^*(\sigma)], \quad \varepsilon_{ij} = \partial W^* / \partial\sigma_{ij} \tag{4.3}$$

Equalities (4.1)–(4.3) constitute a dual formulation of the theory of HMMs which uses the potentials $W(\varepsilon)$ or $W^*(\varepsilon)$. As can be seen from (4.2) and (4.3), this formulation for HMMs (just as in the classical theory of elasticity) includes the Clapeyron formula and the equality of the stress potential and the strain potential.

The principle of virtual work with the usual assumptions [17] also admits of a variational formulation

$$\underline{I} = \inf_{\mathbf{u}} I(\mathbf{u}), \quad I(\mathbf{u}) = \int_V W(\varepsilon(\mathbf{u}))d\upsilon - \int_V \mathbf{F}\mathbf{u}d\upsilon - \int_{S_p} \mathbf{P}\mathbf{u}ds$$
$$\varepsilon(\mathbf{u}) = \tfrac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i) \tag{4.4}$$

where $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$ are the kinematically admissible displacement fields in a Eulerian system of coordinates $\mathbf{x} = (x_1, x_2, x_3)$: $u = u_0$ on a part of the boundary $S_u = S - S_p$ of the domain V occupied by the continuous medium and $\mathbf{F}$ and $\mathbf{P}$ are the densities of the mass and surface forces.

The dual problem to (4.4) is written in the form

$$\bar{I}^* = \sup_\sigma I^*(\sigma), \quad I^*(\sigma) = \int_V W^*(\sigma) d\upsilon - \int_{S_u} (\sigma \mathbf{n}) \mathbf{u}_0 ds \tag{4.5}$$

where σ are the statically possible stress fields: $\partial \sigma_{ij} / \partial x_j + F_i = 0$ in domain V, $\sigma_{ij} n_j = P_i$ on the surface S_p and $\mathbf{n}$ is the unit vector of the outward normal to the small area ds.

By analogy with the case of linear elasticity [17], we have

$$\underline{I} = \bar{I}^*, \quad \inf_{\mathbf{u}} I(\mathbf{u}) = \sup_\sigma I^*(\sigma) \tag{4.6}$$

When account is taken of (4.2) and (4.6), we obtain the Clapeyron theorem

$$2 \int_V W(\varepsilon) d\upsilon = \int_V \mathbf{F} \mathbf{u} d\upsilon + \int_S \mathbf{P} \mathbf{u} ds \tag{4.7}$$

It is also obvious that, as a consequence of (4.2), the extremal problems (4.4) and (4.5) are the Lagrange and Castigliano energy principles for HHMs. Generally speaking, Betti's work reciprocity theorem is not satisfied apart from the case of a proportional change in the stresses at all points of the domain V. This also holds under the assumption of tensor linearity [4].

We note that this expansion of the potential $W(\varepsilon)$ can also be adopted as a basis for developing the model of a granular medium, previously proposed in [18].

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